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Fractional Hermite-Hadamard Integral Inequalities for a New Class of Convex Functions

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Abstract: Fractional integral inequality plays a significant role in pure and applied mathematics fields. It aims to develop and extend various mathematical methods. Therefore, nowadays we need to seek accurate fractional integral inequalities in obtaining the existence and uniqueness of the fractional methods. Besides, the convexity theory plays a concrete role in the field of fractional integral inequalities due to the behavior of its definition and properties. There is also a strong relationship between convexity and symmetric theories. So, whichever one we work on, we can then apply it to the other one due to the strong correlation produced between them, specifically in the last few decades. First, we recall the definition of φ -Riemann–Liouville fractional integral operators and the recently defined class of convex functions, namely the $\check{\sigma}$ -convex functions. Based on these, we will obtain few integral inequalities of Hermite–Hadamard’s type for a $\check{\sigma}$ -convex function with respect to an increasing function involving the φ -Riemann–Liouville fractional integral operator. We can conclude that all derived inequalities in our study generalize numerous well-known inequalities involving both classical and Riemann–Liouville fractional integral inequalities. Finally, application to certain special functions are pointed out.

Keywords: φ -Riemann–Liouville fractional integral; convex functions; $\check{\sigma}$ -convex functions; Hermite–Hadamard inequality

1. Introduction

First of all, we recall the basic notations in convex analysis. A set $\chi \subset \mathbb{R}$ is said to be convex if

$$\eta u + (1 - \eta)v \in \chi,$$

for each $u, v \in \chi$ and $\eta \in [0, 1]$. Based on a convex set χ , we say that a function $Y : \chi \rightarrow \mathbb{R}$ is convex, if the inequality

$$Y(\eta u + (1 - \eta)v) \leq \eta Y(u) + (1 - \eta)Y(v), \quad \eta \in [0, 1] \quad (1)$$

holds for all $u, v \in \chi$. If $-Y$ is convex, then we say that Y is concave.

For the convex functions (1), there are many well-known integral inequalities in the literature; for example, Ostrowski integral inequalities [1], Simpson's integral inequalities [2], Hardy integral inequalities [3], Olsen integral inequalities [4], Gagliardo-Nirenberg integral inequalities [5], Hermite-Hadamard-Fejér integral inequalities [6] and q -Hermite-Hadamard integral inequalities [7]. Also, there is plenty of integral inequalities but the well-known one is the Hermite-Hadamard type integral inequality: the classical and fractional Hermite-Hadamard type integral inequalities [8,9] are, respectively, given by:

$$Y\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v Y(x)dx \leq \frac{Y(u)+Y(v)}{2}, \quad (2)$$

and

$$Y\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\wp+1)}{2(v-u)^\wp} [I_{u^+}^\wp Y(v) + I_{v^-}^\wp Y(u)] \leq \frac{Y(u)+Y(v)}{2}, \quad (3)$$

where $Y: \chi \rightarrow \mathbb{R}$ is assumed to be a positive convex function on χ , $Y \in L^1(u, v)$ with $u < v$, and $I_{u^+}^\wp$ and $I_{v^-}^\wp$ stand for the left-sided and right-sided Riemann-Liouville fractional integrals of order $\wp > 0$, respectively, and these are defined by [10]:

$$\begin{aligned} I_{u^+}^\wp Y(x) &= \frac{1}{\Gamma(\wp)} \int_u^x (x-\eta)^{\wp-1} Y(\eta) d\eta, \quad x > u; \\ I_{v^-}^\wp Y(x) &= \frac{1}{\Gamma(\wp)} \int_x^v (\eta-x)^{\wp-1} Y(\eta) d\eta, \quad x < v. \end{aligned} \quad (4)$$

After introducing Hermite-Hadamard type inequalities (2) and (3), many classical and fractional integral inequalities have been established by a huge number of researcher; for more details one can see References [11–29].

Recently, in Reference [30], Wu et al. introduced a new class of convex sets and convex functions named $\check{\sigma}$ -convex sets and $\check{\sigma}$ -convex functions which are explained in the following definitions:

Definition 1. Let $\check{\sigma} \subseteq \mathbb{R}$ be a strictly monotone and continuous function, and denote $\mathcal{M}_{[\check{\sigma}, \eta]}(u, v) := \check{\sigma}^{-1}(\eta\check{\sigma}(u) + (1-\eta)\check{\sigma}(v))$. We say that a set $\mathcal{Q} \subset \mathbb{R}$ is $\check{\sigma}$ -convex if

$$\mathcal{M}_{[\check{\sigma}, \eta]}(u, v) \in \mathcal{Q}, \quad \eta \in [0, 1] \quad (5)$$

for each $u, v \in \mathcal{Q}$.

Definition 2. A function $Y: \mathcal{Q} \rightarrow \mathbb{R}$ is said to be $\check{\sigma}$ -convex if the inequality

$$Y\left(\mathcal{M}_{[\check{\sigma}, \eta]}(u, v)\right) \leq \eta Y(u) + (1-\eta)Y(v), \quad \eta \in [0, 1] \quad (6)$$

holds for each $u, v \in \mathcal{Q}$.

Remark 1. The function Y is called:

- (i) strictly $\check{\sigma}$ -convex on \mathcal{Q} if (6) is true as a strict inequality for all $\eta \in (0, 1)$ and $u, v \in \mathcal{Q}$ with $u \neq v$.
- (ii) $\check{\sigma}$ -concave on \mathcal{Q} , if $-Y$ is $\check{\sigma}$ -convex on \mathcal{Q} .
- (iii) strictly $\check{\sigma}$ -concave on \mathcal{Q} , if $-Y$ is strictly $\check{\sigma}$ -convex on \mathcal{Q} .

Furthermore, in Reference [30], many inequalities of Hermite-Hadamard's type have been established using the notion of $\check{\sigma}$ -convexity.

The aim of this article is to establish several inequalities of Hermite–Hadamard’s type for $\check{\sigma}$ -convex functions via φ -Riemann–Liouville (φ -RL) fractional integrals, where the φ -RL fractional integrals are defined as follows (see e.g., References [31,32]).

Definition 3. Let $(u, v) \subseteq \mathbb{R}$, $\varphi(x)$ be an increasing and positive function on $(u, v]$ and $\varphi'(x)$ be continuous on (u, v) . Then, the left-sided and right-sided φ -RL fractional integrals of a function Y with respect to the function $\varphi(x)$ on $[u, v]$ are respectively defined by [10,31–33]:

$$\begin{aligned} I_{u+}^{\wp;\varphi} Y(x) &= \frac{1}{\Gamma(\wp)} \int_u^x \varphi'(\eta) (\varphi(x) - \varphi(\eta))^{\wp-1} Y(\eta) d\eta; \\ I_{v-}^{\wp;\varphi} Y(x) &= \frac{1}{\Gamma(\wp)} \int_x^v \varphi'(\eta) (\varphi(\eta) - \varphi(x))^{\wp-1} Y(\eta) d\eta, \quad \wp > 0. \end{aligned} \quad (7)$$

One can observe that if φ is specialized by $\varphi(x) = x$, then φ -Riemann–Liouville fractional integral operators (7) reduce to the classical Riemann–Liouville fractional integral operators (4).

2. Hermite–Hadamard’s Type Inequalities for $\check{\sigma}$ -Convex Functions

Our main results depend on the following lemmas:

Theorem 1. Assume that the function $Y : [u, v] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is integrable $\check{\sigma}$ -convex and $Y \in L^1(u, v)$ with $0 \leq u < v$. If the function $\check{\sigma}$ is increasing and positive on $(u, v]$ and $\check{\sigma}'(x)$ is continuous on (u, v) . Then, we have for $\wp > 0$:

$$Y \left(\check{\sigma}^{-1} \left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2} \right) \right) \leq \frac{\Gamma(\wp + 1)}{2(\check{\sigma}(v) - \check{\sigma}(u))^\wp} \left[I_{u+}^{\wp;\check{\sigma}} Y(v) + I_{v-}^{\wp;\check{\sigma}} Y(u) \right] \leq \frac{Y(u) + Y(v)}{2}. \quad (8)$$

Proof. Since Y is a $\check{\sigma}$ -convex function, we have

$$Y \left(\check{\sigma}^{-1} \left(\frac{\check{\sigma}(x) + \check{\sigma}(y)}{2} \right) \right) \leq \frac{Y(x) + Y(y)}{2}. \quad (9)$$

Substituting $x = \check{\sigma}^{-1}(\eta\check{\sigma}(u) + (1 - \eta)\check{\sigma}(v))$ and $y = \check{\sigma}^{-1}((1 - \eta)\check{\sigma}(u) + \eta\check{\sigma}(v))$ into (9), we get

$$2Y \left(\check{\sigma}^{-1} \left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2} \right) \right) \leq Y \left(\check{\sigma}^{-1}(\eta\check{\sigma}(u) + (1 - \eta)\check{\sigma}(v)) \right) + Y \left(\check{\sigma}^{-1}((1 - \eta)\check{\sigma}(u) + \eta\check{\sigma}(v)) \right). \quad (10)$$

Multiplying both sides of (10) by $\eta^{\wp-1}$, then integrating the resulting inequality with respect to η over $[0, 1]$, we get

$$\begin{aligned} \frac{2}{\wp} Y \left(\check{\sigma}^{-1} \left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2} \right) \right) &\leq \int_0^1 \eta^{\wp-1} Y \left(\check{\sigma}^{-1}(\eta\check{\sigma}(u) + (1 - \eta)\check{\sigma}(v)) \right) d\eta \\ &\quad + \int_0^1 \eta^{\wp-1} Y \left(\check{\sigma}^{-1}((1 - \eta)\check{\sigma}(u) + \eta\check{\sigma}(v)) \right) d\eta. \end{aligned}$$

By changing the variables $z = \check{\sigma}^{-1}(\eta\check{\sigma}(u) + (1 - \eta)\check{\sigma}(v))$ and $w = \check{\sigma}^{-1}((1 - \eta)\check{\sigma}(u) + \eta\check{\sigma}(v))$, then the last inequality becomes

$$\begin{aligned} Y \left(\check{\sigma}^{-1} \left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2} \right) \right) &\leq \frac{\wp}{2(\check{\sigma}(v) - \check{\sigma}(u))^\wp} \int_u^v \check{\sigma}'(z) (\check{\sigma}(v) - \check{\sigma}(z))^{\wp-1} Y(z) dz \\ &\quad + \frac{\wp}{2(\check{\sigma}(v) - \check{\sigma}(u))^\wp} \int_u^v \check{\sigma}'(w) (\check{\sigma}(w) - \check{\sigma}(u))^{\wp-1} Y(w) dw \\ &= \frac{\Gamma(\wp + 1)}{2(\check{\sigma}(v) - \check{\sigma}(u))^\wp} \left[I_{u+}^{\wp;\check{\sigma}} Y(v) + I_{v-}^{\wp;\check{\sigma}} Y(u) \right]. \end{aligned}$$

This completes the proof of our first inequality in (8).

In order to prove the second inequality in (8), we use the $\check{\sigma}$ -convexity of Y ; that is

$$Y\left(\check{\sigma}^{-1}(\eta\check{\sigma}(u) + (1-\eta)\check{\sigma}(v))\right) \leq \eta Y(u) + (1-\eta)Y(v),$$

and

$$Y\left(\check{\sigma}^{-1}((1-\eta)\check{\sigma}(u) + \eta\check{\sigma}(v))\right) \leq (1-\eta)Y(u) + \eta Y(v).$$

By adding these two inequalities we get

$$\begin{aligned} Y\left(\check{\sigma}^{-1}(\eta\check{\sigma}(u) + (1-\eta)\check{\sigma}(v))\right) + Y\left(\check{\sigma}^{-1}((1-\eta)\check{\sigma}(u) + \eta\check{\sigma}(v))\right) \\ \leq \eta Y(u) + (1-\eta)Y(v) + (1-\eta)Y(u) + \eta Y(v). \end{aligned} \quad (11)$$

Multiplying both sides of (11) by $\eta^{\wp-1}$ and then integrating with respect to η over $[0, 1]$, we can obtain

$$\begin{aligned} \int_0^1 \eta^{\wp-1} Y\left(\check{\sigma}^{-1}(\eta\check{\sigma}(u) + (1-\eta)\check{\sigma}(v))\right) d\eta \\ + \int_0^1 \eta^{\wp-1} Y\left(\check{\sigma}^{-1}((1-\eta)\check{\sigma}(u) + \eta\check{\sigma}(v))\right) d\eta \leq (Y(u) + Y(v)) \int_0^1 \eta^{\wp-1} d\eta, \end{aligned}$$

hence,

$$\frac{\Gamma(\wp+1)}{2(\check{\sigma}(v) - \check{\sigma}(u))^{\wp}} \left[I_{u^+}^{\wp;\check{\sigma}} Y(v) + I_{v^-}^{\wp;\check{\sigma}} Y(u) \right] \leq \frac{Y(u) + Y(v)}{2}.$$

This completes the proof of our Lemma 1. \square

Remark 2. Particularly, in Lemma 1, if we take

- (i) $\check{\sigma}(x) = x$, then inequality (8) reduces to inequality (3).
- (ii) $\wp = 1$, then we have

$$Y\left(\check{\sigma}^{-1}\left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2}\right)\right) \leq \frac{1}{\check{\sigma}(v) - \check{\sigma}(u)} \int_u^v Y(x) \check{\sigma}'(x) dx \leq \frac{Y(u) + Y(v)}{2},$$

which was already established in Reference [30].

- (iii) $\check{\sigma}(x) = x$ and $\wp = 1$, then inequality (8) reduces to inequality (2).

Theorem 2. Assume that the function $Y : [u, v] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is integrable $\check{\sigma}$ -convex and $Y \in L^1(u, v)$ with $0 \leq u < v$. If the function $\check{\sigma}$ is increasing and positive on $(u, v]$ and $\check{\sigma}'(x)$ is continuous on (u, v) . Then, we have for $\wp > 0$:

$$\begin{aligned} Y\left(\check{\sigma}^{-1}\left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2}\right)\right) \leq \frac{\Gamma(\wp+1)2^{\wp-1}}{(\check{\sigma}(v) - \check{\sigma}(u))^{\wp}} \left[I_{\left(\check{\sigma}^{-1}\left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2}\right)\right)^+}^{\wp;\check{\sigma}} Y(v) + I_{\left(\check{\sigma}^{-1}\left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2}\right)\right)^-}^{\wp;\check{\sigma}} Y(u) \right] \\ \leq \frac{Y(u) + Y(v)}{2}. \end{aligned} \quad (12)$$

Proof. Again, since Y is a $\check{\sigma}$ -convex function, we can substituting $x = \check{\sigma}^{-1} \left(\frac{\eta}{2} \check{\sigma}(u) + \frac{2-\eta}{2} \check{\sigma}(v) \right)$ and $y = \check{\sigma}^{-1} \left(\frac{2-\eta}{2} \check{\sigma}(u) + \frac{\eta}{2} \check{\sigma}(v) \right)$ into (9) to get

$$2Y \left(\check{\sigma}^{-1} \left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2} \right) \right) \leq Y \left(\check{\sigma}^{-1} \left(\frac{\eta}{2} \check{\sigma}(u) + \frac{2-\eta}{2} \check{\sigma}(v) \right) \right) + Y \left(\check{\sigma}^{-1} \left(\frac{2-\eta}{2} \check{\sigma}(u) + \frac{\eta}{2} \check{\sigma}(v) \right) \right). \quad (13)$$

Multiplying both sides of (13) by $\eta^{\wp-1}$, then integrating the resulting inequality with respect to η over $[0, 1]$, we get

$$\frac{2}{\wp} Y \left(\check{\sigma}^{-1} \left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2} \right) \right) \leq \int_0^1 \eta^{\wp-1} Y \left(\check{\sigma}^{-1} \left(\frac{\eta}{2} \check{\sigma}(u) + \frac{2-\eta}{2} \check{\sigma}(v) \right) \right) d\eta + \int_0^1 \eta^{\wp-1} Y \left(\check{\sigma}^{-1} \left(\frac{2-\eta}{2} \check{\sigma}(u) + \frac{\eta}{2} \check{\sigma}(v) \right) \right) d\eta.$$

By changing the variables $z = \check{\sigma}^{-1} \left(\frac{\eta}{2} \check{\sigma}(u) + \frac{2-\eta}{2} \check{\sigma}(v) \right)$ and $w = \check{\sigma}^{-1} \left(\frac{2-\eta}{2} \check{\sigma}(u) + \frac{\eta}{2} \check{\sigma}(v) \right)$, then the last inequality becomes

$$\begin{aligned} Y \left(\check{\sigma}^{-1} \left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2} \right) \right) &\leq \frac{\wp 2^{\wp-1}}{(\check{\sigma}(v) - \check{\sigma}(u))^{\wp}} \int_{\check{\sigma}^{-1} \left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2} \right)}^v \check{\sigma}'(z) (\check{\sigma}(v) - \check{\sigma}(z))^{\wp-1} Y(z) dz \\ &\quad + \frac{\wp 2^{\wp-1}}{(\check{\sigma}(v) - \check{\sigma}(u))^{\wp}} \int_u^{\check{\sigma}^{-1} \left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2} \right)} \check{\sigma}'(w) (\check{\sigma}(w) - \check{\sigma}(u))^{\wp-1} Y(w) dw \\ &= \frac{\Gamma(\wp + 1) 2^{\wp-1}}{(\check{\sigma}(v) - \check{\sigma}(u))^{\wp}} \left[I_{\left(\check{\sigma}^{-1} \left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2} \right) \right)^+}^{\wp, \check{\sigma}} Y(v) + I_{\left(\check{\sigma}^{-1} \left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2} \right) \right)^-}^{\wp, \check{\sigma}} Y(u) \right]. \end{aligned}$$

This completes the proof of our first inequality in (12).

In order to obtain the second inequality in (12), we use the $\check{\sigma}$ -convexity of Y as:

$$Y \left(\check{\sigma}^{-1} \left(\frac{\eta}{2} \check{\sigma}(u) + \frac{2-\eta}{2} \check{\sigma}(v) \right) \right) \leq \frac{\eta}{2} Y(u) + \frac{2-\eta}{2} Y(v),$$

and

$$Y \left(\check{\sigma}^{-1} \left(\frac{2-\eta}{2} \check{\sigma}(u) + \frac{\eta}{2} \check{\sigma}(v) \right) \right) \leq \frac{2-\eta}{2} Y(u) + \frac{\eta}{2} Y(v).$$

By adding these two inequalities we get

$$Y \left(\check{\sigma}^{-1} \left(\frac{\eta}{2} \check{\sigma}(u) + \frac{2-\eta}{2} \check{\sigma}(v) \right) \right) + Y \left(\check{\sigma}^{-1} \left(\frac{2-\eta}{2} \check{\sigma}(u) + \frac{\eta}{2} \check{\sigma}(v) \right) \right) \leq Y(u) + Y(v). \quad (14)$$

Multiplying both sides of (14) by $\eta^{\wp-1}$ and then integrating with respect to η over $[0, 1]$, we get

$$\begin{aligned} \int_0^1 \eta^{\wp-1} Y \left(\check{\sigma}^{-1} \left(\frac{\eta}{2} \check{\sigma}(u) + \frac{2-\eta}{2} \check{\sigma}(v) \right) \right) d\eta \\ + \int_0^1 \eta^{\wp-1} Y \left(\check{\sigma}^{-1} \left(\frac{2-\eta}{2} \check{\sigma}(u) + \frac{\eta}{2} \check{\sigma}(v) \right) \right) d\eta \leq (Y(u) + Y(v)) \int_0^1 \eta^{\wp-1} d\eta, \end{aligned}$$

namely,

$$\frac{\Gamma(\wp+1)2^{\wp-1}}{(\check{\sigma}(v)-\check{\sigma}(u))^{\wp}} \left[I_{(\check{\sigma}^{-1}(\frac{\check{\sigma}(u)+\check{\sigma}(v)}{2}))^{+}}^{\wp;\check{\sigma}} Y(v) + I_{(\check{\sigma}^{-1}(\frac{\check{\sigma}(u)+\check{\sigma}(v)}{2}))^{-}}^{\wp;\check{\sigma}} Y(u) \right] \leq \frac{Y(u)+Y(v)}{2}.$$

This completes the proof of our Lemma 2. \square

Remark 3. Especially, in Lemma 2, if we take

(i) $\check{\sigma}(x) = x$, then inequality (12) reduces to the following inequality:

$$Y\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\wp+1)2^{\wp-1}}{(v-u)^{\wp}} \left[I_{(\frac{u+v}{2})^{+}}^{\wp} Y(v) + I_{(\frac{u+v}{2})^{-}}^{\wp} Y(u) \right] \leq \frac{Y(u)+Y(v)}{2},$$

which was already established in Reference [34].

(ii) $\wp = 1$, then it holds

$$Y\left(\check{\sigma}^{-1}\left(\frac{\check{\sigma}(u)+\check{\sigma}(v)}{2}\right)\right) \leq \frac{1}{\check{\sigma}(v)-\check{\sigma}(u)} \int_u^v Y(x)\check{\sigma}'(x)dx \leq \frac{Y(u)+Y(v)}{2},$$

which was already established in Reference [30].

(iii) $\check{\sigma}(x) = x$ and $\wp = 1$, then inequality (12) reduces to inequality (2).

3. Further Consequences

As consequences for the Lemmas 1 and 2, we can obtain the following theorems.

Theorem 3. Assume that $Y : [u, v] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is an L^1 integrable $\check{\sigma}$ -convex function and $Y' \in L^1(u, v)$ with $0 \leq u < v$. If the function $\check{\sigma}$ is increasing and positive on $(u, v]$ and $\check{\sigma}'(x)$ is continuous on (u, v) . Then, we have for $\wp > 0$:

$$\begin{aligned} \frac{Y(u)+Y(v)}{2} - \frac{\Gamma(\wp+1)}{2(\check{\sigma}(v)-\check{\sigma}(u))^{\wp}} \left[I_{u^{+}}^{\wp;\check{\sigma}} Y(v) + I_{v^{-}}^{\wp;\check{\sigma}} Y(u) \right] \\ = \frac{1}{2[\check{\sigma}(v)-\check{\sigma}(u)]^{\wp}} \int_u^v ((\check{\sigma}(z)-\check{\sigma}(u))^{\wp} - (\check{\sigma}(v)-\check{\sigma}(z))^{\wp}) Y'(z) dz. \end{aligned} \quad (15)$$

Proof. By Definition 3 and integrating by parts one can find

$$\begin{aligned} I_1 &:= \frac{\Gamma(\wp+1)}{2(\check{\sigma}(v)-\check{\sigma}(u))^{\wp}} I_{u^{+}}^{\wp;\check{\sigma}} Y(v) = \frac{\wp}{2[\check{\sigma}(v)-\check{\sigma}(u)]^{\wp}} \int_u^v \check{\sigma}'(z) [\check{\sigma}(v)-\check{\sigma}(z)]^{\wp-1} Y(z) dz \\ &= \frac{-1}{2[\check{\sigma}(v)-\check{\sigma}(u)]^{\wp}} \int_u^v Y(z) d(\check{\sigma}(v)-\check{\sigma}(z))^{\wp} \\ &= \frac{1}{2[\check{\sigma}(v)-\check{\sigma}(u)]^{\wp}} \left[[\check{\sigma}(v)-\check{\sigma}(u)]^{\wp} Y(u) + \int_u^v [\check{\sigma}(v)-\check{\sigma}(z)]^{\wp} Y'(z) dz \right]. \end{aligned} \quad (16)$$

Analogously, we get

$$\begin{aligned} I_2 &:= \frac{\Gamma(\wp+1)}{2[\check{\sigma}(v)-\check{\sigma}(u)]^{\wp}} I_{v^{-}}^{\wp;\check{\sigma}} Y(u) = \frac{\wp}{2[\check{\sigma}(v)-\check{\sigma}(u)]^{\wp}} \int_u^v \check{\sigma}'(z) [\check{\sigma}(z)-\check{\sigma}(u)]^{\wp-1} Y(z) dz \\ &= \frac{1}{2[\check{\sigma}(v)-\check{\sigma}(u)]^{\wp}} \left[[\check{\sigma}(v)-\check{\sigma}(u)]^{\wp} Y(v) - \int_u^v [\check{\sigma}(z)-\check{\sigma}(u)]^{\wp} Y'(z) dz \right]. \end{aligned} \quad (17)$$

From the identities (16) and (17), it yields

$$\frac{Y(u) + Y(v)}{2} - (I_1 + I_2) = \frac{1}{2[\check{\sigma}(v) - \check{\sigma}(u)]^\varphi} \int_u^v ((\check{\sigma}(z) - \check{\sigma}(u))^\varphi - (\check{\sigma}(v) - \check{\sigma}(z))^\varphi) Y'(z) dz.$$

This completes the proof of Theorem 3. \square

As a particular case of Theorem 3, if $\check{\sigma}$ is specialized by $\check{\sigma}(z) = z$, then we have the following corollary, which has been studied by Sarikaya et al. in Reference [9].

Corollary 1. Under the same assumptions of Theorem 3, if $\check{\sigma}(z) = z$, then we have

$$\frac{Y(u) + Y(v)}{2} - \frac{\Gamma(\varphi + 1)}{2(v - u)^\varphi} [I_{u^+}^\varphi Y(v) + I_{v^-}^\varphi Y(u)] = \frac{1}{2(v - u)^\varphi} \int_u^v [(z - u)^\varphi - (v - z)^\varphi] Y'(z) dz. \quad (18)$$

Moreover, if $|Y'|$ is convex on $[u, v]$, then we have

$$\left| \frac{Y(u) + Y(v)}{2} - \frac{\Gamma(\varphi + 1)}{2(v - u)^\varphi} [I_{u^+}^\varphi Y(v) + I_{v^-}^\varphi Y(u)] \right| \leq \frac{v - u}{2(\varphi + 1)} \left(1 - \frac{1}{2^\varphi} \right) [|Y'(u)| + |Y'(v)|]. \quad (19)$$

Proof. By putting $\check{\sigma}(z) = z$ into Theorem 3, we directly obtain the desired equality (18). To prove the inequality (19), we change the variable $z = (1 - \eta)u + \eta v$ in (18), we have

$$\frac{Y(u) + Y(v)}{2} - \frac{\Gamma(\varphi + 1)}{2(v - u)^\varphi} [I_{u^+}^\varphi Y(v) + I_{v^-}^\varphi Y(u)] = \frac{v - u}{2} \int_0^1 [(1 - \eta)^\varphi - \eta^\varphi] Y'((1 - \eta)u + \eta v) d\eta.$$

Then, we obtain (19) as in the proof of [9, Theorem 3]. \square

Theorem 4. Assume that $Y : [u, v] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is an L^1 integrable $\check{\sigma}$ -convex function and $Y' \in L^1(u, v)$ with $0 \leq u < v$. If the function $\check{\sigma}$ is increasing and positive on $(u, v]$ and $\check{\sigma}'(x)$ is continuous on (u, v) . Then, we have

$$\begin{aligned} & \frac{\Gamma(\varphi + 1)2^{\varphi-1}}{(\check{\sigma}(v) - \check{\sigma}(u))^\varphi} \left[I_{\left(\check{\sigma}^{-1}\left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2}\right)\right)^+}^{\varphi; \check{\sigma}} Y(v) + I_{\left(\check{\sigma}^{-1}\left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2}\right)\right)^-}^{\varphi; \check{\sigma}} Y(u) \right] - Y\left(\check{\sigma}^{-1}\left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2}\right)\right) \\ &= \frac{2^{\varphi-1}}{(\check{\sigma}(v) - \check{\sigma}(u))^\varphi} \left(\int_{\check{\sigma}^{-1}\left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2}\right)}^v (\check{\sigma}(v) - \check{\sigma}(z))^\varphi Y'(z) dz \right. \\ & \quad \left. - \int_u^{\check{\sigma}^{-1}\left(\frac{\check{\sigma}(u) + \check{\sigma}(v)}{2}\right)} (\check{\sigma}(z) - \check{\sigma}(u))^\varphi Y'(z) dz \right), \quad (20) \end{aligned}$$

for $\varphi > 0$.

Proof. By Definition 3 and integrating by parts one can find

$$\begin{aligned}
 J_1 &:= \frac{\Gamma(\wp+1)2^{\wp-1}}{(\wp(v)-\wp(u))^{\wp}} I_{\left(\wp^{-1}\left(\frac{\wp(u)+\wp(v)}{2}\right)\right)^+}^{\wp;\wp} Y(v) \\
 &= \frac{\wp 2^{\wp-1}}{(\wp(v)-\wp(u))^{\wp}} \int_{\wp^{-1}\left(\frac{\wp(u)+\wp(v)}{2}\right)}^v \wp'(z)(\wp(v)-\wp(z))^{\wp-1} Y(z) dz \\
 &= \frac{-2^{\wp-1}}{(\wp(v)-\wp(u))^{\wp}} \int_{\wp^{-1}\left(\frac{\wp(u)+\wp(v)}{2}\right)}^v Y(z) d(\wp(v)-\wp(z))^{\wp} \\
 &= \frac{2^{\wp-1}}{(\wp(v)-\wp(u))^{\wp}} \left[\left(\frac{\wp(v)-\wp(u)}{2} \right)^{\wp} Y\left(\wp^{-1}\left(\frac{\wp(u)+\wp(v)}{2}\right)\right) + \int_{\wp^{-1}\left(\frac{\wp(u)+\wp(v)}{2}\right)}^v (\wp(v)-\wp(z))^{\wp} Y'(z) dz \right] \\
 &= \frac{1}{2} Y\left(\wp^{-1}\left(\frac{\wp(u)+\wp(v)}{2}\right)\right) + \frac{2^{\wp-1}}{(\wp(v)-\wp(u))^{\wp}} \int_{\wp^{-1}\left(\frac{\wp(u)+\wp(v)}{2}\right)}^v (\wp(v)-\wp(z))^{\wp} Y'(z) dz. \quad (21)
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 J_2 &:= \frac{\Gamma(\wp+1)2^{\wp-1}}{(\wp(v)-\wp(u))^{\wp}} I_{\left(\wp^{-1}\left(\frac{\wp(u)+\wp(v)}{2}\right)\right)^-}^{\wp;\wp} Y(u) \\
 &= \frac{\wp 2^{\wp-1}}{(\wp(v)-\wp(u))^{\wp}} \int_u^{\wp^{-1}\left(\frac{\wp(u)+\wp(v)}{2}\right)} \wp'(z)(\wp(z)-\wp(u))^{\wp-1} Y(z) dz \\
 &= \frac{1}{2} Y\left(\wp^{-1}\left(\frac{\wp(u)+\wp(v)}{2}\right)\right) - \frac{2^{\wp-1}}{(\wp(v)-\wp(u))^{\wp}} \int_u^{\wp^{-1}\left(\frac{\wp(u)+\wp(v)}{2}\right)} (\wp(z)-\wp(u))^{\wp} Y'(z) dz. \quad (22)
 \end{aligned}$$

From the identities (21) and (22), one has

$$\begin{aligned}
 J_1 + J_2 - Y\left(\wp^{-1}\left(\frac{\wp(u)+\wp(v)}{2}\right)\right) \\
 = \frac{2^{\wp-1}}{(\wp(v)-\wp(u))^{\wp}} \left(\int_{\wp^{-1}\left(\frac{\wp(u)+\wp(v)}{2}\right)}^v (\wp(v)-\wp(z))^{\wp} Y'(z) dz - \int_u^{\wp^{-1}\left(\frac{\wp(u)+\wp(v)}{2}\right)} (\wp(z)-\wp(u))^{\wp} Y'(z) dz \right).
 \end{aligned}$$

This ends the proof of our Theorem 4. \square

Additionally, when $\wp(z) = z$, then our result Theorem 4 becomes to the following corollary, which has been already explored by Sarikaya et al. in [34].

Corollary 2. *Particularly, if we take $\wp(z) = z$ in Theorem 4, we get*

$$\begin{aligned}
 \frac{\Gamma(\wp+1)2^{\wp-1}}{(v-u)^{\wp}} \left[I_{\left(\frac{u+v}{2}\right)^+}^{\wp} Y(v) + I_{\left(\frac{u+v}{2}\right)^-}^{\wp} Y(u) \right] - Y\left(\frac{u+v}{2}\right) \\
 = \frac{2^{\wp-1}}{(v-u)^{\wp}} \left(\int_{\frac{u+v}{2}}^v (v-z)^{\wp} Y'(z) dz - \int_u^{\frac{u+v}{2}} (z-u)^{\wp} Y'(z) dz \right). \quad (23)
 \end{aligned}$$

Moreover, if $|Y'|^q, q \geq 1$ is convex on $[u, v]$, we have

$$\begin{aligned}
 \left| \frac{\Gamma(\wp+1)2^{\wp-1}}{(v-u)^{\wp}} \left[I_{\left(\frac{u+v}{2}\right)^+}^{\wp} Y(v) + I_{\left(\frac{u+v}{2}\right)^-}^{\wp} Y(u) \right] - Y\left(\frac{u+v}{2}\right) \right| \\
 \leq \frac{v-u}{4(\wp+1)} \left(\frac{1}{2(\wp+2)} \right)^{\frac{1}{q}} \left[((\wp+1)|Y'(u)|^q + (\wp+3)|Y'(v)|^q)^{\frac{1}{q}} \right. \\
 \left. + ((\wp+3)|Y'(u)|^q + (\wp+1)|Y'(v)|^q)^{\frac{1}{q}} \right]. \quad (24)
 \end{aligned}$$

Proof. By putting $\check{\sigma}(z) = z$ into Theorem 4, we directly obtain the desired equality (23). To prove the inequality (24), we use the same proof technique of Corollary 1 with the convexity of $|Y'|^q$. Thus, the proof of Corollary 2 is completed. \square

4. Applications

4.1. The Modified Bessel Functions

Consider the function $\mathcal{W}_\varsigma : \mathbb{R} \rightarrow [1, \infty)$, defined by:

$$\mathcal{W}_\varsigma(z) = 2^\varsigma \Gamma(\varsigma + 1) z^{-\varsigma} S_\varsigma(z), \quad z \in \mathbb{R}, \varsigma > -1.$$

Then, the modified Bessel function of the first S_ς and the second kind \mathcal{Y}_ς are defined as follows [35]:

$$S_\varsigma(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\varsigma+2n}}{n! \Gamma(\varsigma + n + 1)};$$

$$\mathcal{Y}_\varsigma(z) = \frac{\pi}{2} \frac{S_{-\varsigma}(z) + S_\varsigma(z)}{\sin(\varsigma\pi)}.$$

respectively.

Proposition 1. Assume that $u, v \in \mathbb{R}$ with $0 < u < v$. Then, we have for each $\varsigma > -1$:

$$\left| \frac{(\mathcal{W}_\varsigma \circ \check{\sigma})(v) - (\mathcal{W}_\varsigma \circ \check{\sigma})(u)}{\check{\sigma}(v) - \check{\sigma}(u)} \right| \leq \frac{\check{\sigma}(u) (\mathcal{W}_{\varsigma+1} \circ \check{\sigma})(u) + \check{\sigma}(v) (\mathcal{W}_{\varsigma+1} \circ \check{\sigma})(v)}{4(\varsigma + 1)}, \quad (25)$$

where $\check{\sigma}$ as before.

Proof. Following Reference [35], we have

$$\mathcal{W}'_\varsigma(z) = \frac{z}{2(\varsigma + 1)} \mathcal{W}_{\varsigma+1}(z). \quad (26)$$

One can observe that the function $\mathcal{W}'_\varsigma(z)$ is convex on $(0, \infty)$ for $\varsigma > -1$. So, the function $Y(z) := (\mathcal{W}'_\varsigma \circ \check{\sigma})(z)$ is $\check{\sigma}$ -convex on the same interval. Using (26) and Lemma 1 with $\wp = 1$, we obtain the desired inequality (25) immediately. \square

Proposition 2. Assume that $u, v \in \mathbb{R}$ with $0 < u < v$. Then, we have for each $\varsigma > -1$:

$$\left| \frac{\check{\sigma}^\varsigma(u) (\mathcal{Y}_\varsigma \circ \check{\sigma})(v) - \check{\sigma}^\varsigma(v) (\mathcal{Y}_\varsigma \circ \check{\sigma})(u)}{\check{\sigma}(v) - \check{\sigma}(u)} \right| \leq \frac{\check{\sigma}^\varsigma(v) (\mathcal{Y}_{\varsigma+1} \circ \check{\sigma})(u) + \check{\sigma}^\varsigma(u) (\mathcal{Y}_{\varsigma+1} \circ \check{\sigma})(v)}{2}, \quad (27)$$

where $\check{\sigma}$ as before.

Proof. Let $f_\varsigma(z) := \frac{\mathcal{Y}_\varsigma(z)}{z^\varsigma}$, where $\varsigma \in \mathbb{R}$. Consider the integral formula [35]:

$$\mathcal{Y}_\varsigma(z) = \int_0^\infty e^{-z \cosh(\eta)} \cosh(\varsigma\eta) d\eta, \quad z > 0.$$

One can note that the function $z \mapsto \mathcal{Y}_\varsigma(z)$ is completely monotonic on $(0, \infty)$ for any $\varsigma \in \mathbb{R}$. Then, we conclude that $z \mapsto f_\varsigma(z)$ is strictly complete monotonic on $(0, \infty)$ for each $\varsigma > -1$ since the product of two strictly completely monotonic functions is strictly completely monotonic as well. Therefore, the function $-f'_\varsigma(z) = -\left(\frac{\mathcal{Y}_\varsigma(z)}{z^\varsigma}\right)' = -\frac{\mathcal{Y}_{\varsigma+1}(z)}{z^\varsigma}$ is strictly completely monotonic on $(0, \infty)$ for any $\varsigma > -1$ and thus f_ς is convex.

Now, setting $Y(z) := -\left(f'_\varsigma \circ \check{\sigma}\right)(z)$, then one can conclude that the function $Y(z)$ is $\check{\sigma}$ -convex. Consequently, by applying Lemma 1 with above $Y(z)$ and $\wp = 1$, we can obtain the desired inequality (27) immediately for $\varsigma > -1$. \square

4.2. Special Means

We consider the special means of positive numbers $u \neq v$:

- The arithmetic mean:

$$A(u, v) = \frac{u + v}{2}.$$

- The generalized logarithmic mean:

$$L_\varsigma(u, v) = \left[\frac{v^{\varsigma+1} - u^{\varsigma+1}}{(\varsigma+1)(v-u)} \right]^{\frac{1}{\varsigma}}, \quad \varsigma \in \mathbb{R} \setminus \{-1, 0\}.$$

Proposition 3. Let $0 < u < v$, where $u, v \in \mathbb{R}$. Then, we have

$$\ln^2 [A(e^u, e^v)] \leq v^2 e^v - u^2 e^u - 4A((1-u)e^u, (v-1)e^v) \leq A(u^2, v^2). \quad (28)$$

Proof. Taking $Y(z) = z^2$, $\check{\sigma}(z) = e^z$ and $\wp = 1$ in Remark 2 part (ii), the double inequality (28) is obtained. \square

Proposition 4. Let $0 < u < v$, where $u, v \in \mathbb{R}$. Then for $\varsigma > 0$, we get

$$A^{\frac{2}{\varsigma}}(u^\varsigma, v^\varsigma) \leq \frac{\varsigma(v-u)}{v^\varsigma - u^\varsigma} L_{\varsigma+1}^{\varsigma+1}(u, v) \leq A(u^2, v^2). \quad (29)$$

Proof. Choosing $Y(z) = z^2$, $\check{\sigma}(z) = z^\varsigma$ and $\wp = 1$ in Remark 2 part (ii), the double inequality (29) is captured. \square

5. Conclusions

In the study, we have considered a new class of convex functions and the definition of φ -RL fractional integral operators. In our present investigation, we have established new fractional Hermite-Hadamard's integral inequalities associated to increasing functions. The results obtained here are very useful in obtaining other type of inequalities. Also, these results are very generic and can be specified to give further potentially useful and interesting integral inequalities involving other type of fractional integral operators.

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